

Index computations on FJRW theory

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Abstract

We compute the index of the real Cauchy-Riemann operator defined in FJRW theory in case of the smooth metric. For the cylindrical metric, we study the relation between the index of the linearized operator of Witten map and weights in weighted Sobolev space.

1 Introduction

In this paper we mainly investigate two index problems coming from *FJRW* theory (see [10, 11, 12]) constructed by Huijun Fan, Tyler J. Jarvis, Yongbin Ruan based on a proposal of Edward Witten.

One is a concrete index computation problem of the real Cauchy-Riemann operator under the smooth metric, the other is about relation between the index of the linearized operator of Witten map [12] and weights in weighted Sobolev space in case of the cylindrical metric. They are both interesting and also important for *FJRW* theory.

First of all let us review the background of these problems. We adopt notations of [10] in the following discussion.

Let $W \in \mathbb{C}[x_1, \dots, x_t]$ be a quasi-homogeneous polynomial, i.e., there exists degrees $d, k_1, \dots, k_t \in \mathbb{Z}^{>0}$ such that for any $\lambda \in \mathbb{C}^*$

$$W(\lambda^{k_1}x_1, \dots, \lambda^{k_t}x_t) = \lambda^d W(x_1, \dots, x_t). \quad (1)$$

Definition 1.1. W is called *nondegenerate* if

- (1) the fractional degrees $q_i = \frac{k_i}{d}$ are uniquely determined by W ; and
- (2) the hypersurface defined by W in weighted projective space is non-singular, or equivalently, the affine hypersurface defined by W has an isolated singularity at the origin.

Lemma 1.1. ([10]) *If W is nondegenerate, then the group*

$$H := \{(\alpha_1, \dots, \alpha_t) \in (\mathbb{C}^*)^t \mid W(\alpha_1 x_1, \dots, \alpha_t x_t) = W(x_1, \dots, x_t)\} \quad (2)$$

of diagonal symmetries of W is finite. In particular, we have

$$H \subseteq \mu_{d/k_1} \times \dots \times \mu_{d/k_t} \cong k_1 \mathbb{Z}/d \times \dots \times k_t \mathbb{Z}/d, \quad (3)$$

where μ_l is the group of l th roots of unity.

Here finiteness of the group H is necessary for later discussion.

W-spin structures on smooth orbicurves. Let $(\tilde{\Sigma}, \mathbf{z}, \mathbf{m})$ be a smooth orbicurve (orbifold Riemann surface), i.e., $(\tilde{\Sigma}, \mathbf{z}, \mathbf{m})$ is a Riemann surface Σ with marked points $\mathbf{z} = \{z_i\}$ having orbifold structure near each marked point z_i given by a faithful action of \mathbb{Z}/m_i .

In other words, a neighborhood of each marked point is uniformized by the branched covering map $z \rightarrow z^{m_i}$.

Let $\rho : \tilde{\Sigma} \rightarrow \Sigma$ be the natural projection to the coarse Riemann surface Σ . A line bundle L on Σ can be uniquely lifted to an orbifold line bundle on $\tilde{\Sigma}$. We denote the lifted bundle by the same L .

Definition 1.2. Let K be the canonical bundle of Σ , and let

$$K_{log} := K \otimes \mathcal{O}(z_1) \otimes \cdots \otimes \mathcal{O}(z_k) \quad (4)$$

be the *log-canonical bundle*. The holomorphic sections are holomorphic 1-forms away from the special points $\{z_i\}$ and with at simple poles at the z_i . K_{log} can be thought of as the canonical bundle of the punctured Riemann surface $\Sigma - \{z_1, \dots, z_k\}$. Suppose that L_1, \dots, L_t are orbifold line bundles on $\tilde{\Sigma}$ with isomorphisms $\varphi_j : W_j(L_1, \dots, L_t) \xrightarrow{\sim} K_{log}$, where by $W_j(L_1, \dots, L_t)$ we mean the j th monomial of W in L_i ,

$$W_j(L_1, \dots, L_t) = L_1^{\otimes b_{1j}} \otimes \cdots \otimes L_t^{\otimes b_{tj}}. \quad (5)$$

Here K_{log} is identified with its pull-back to $\tilde{\Sigma}$.

The tuple $(L_1, \dots, L_t, \varphi_1, \dots, \varphi_s)$ is called a *W-spin structure*.

Definition 1.3. Suppose that the chart of $\tilde{\Sigma}$ at an orbifold point z_i is $D/(\mathbb{Z}/m)$ with action $e^{\frac{2\pi i}{m}}(z) = e^{\frac{2\pi i}{m}}z$. Suppose that the local trivialization of an orbifold line bundle L is $(D \times \mathbb{C})/(\mathbb{Z}/m)$ with the action

$$e^{\frac{2\pi i}{m}}(z, w) = (e^{\frac{2\pi i}{m}}z, e^{\frac{2\pi i\nu}{m}}w). \quad (6)$$

When $\nu = 0$, we say that L is *Broad* at z_i ; when $\nu > 0$, we say L is *Narrow* at z_i .

A *W-spin structure* $(L_1, \dots, L_t, \varphi_1, \dots, \varphi_s)$ is called *Broad* at the point z_i if the group element $h = (\exp(2\pi i\nu_1/m), \dots, \exp(2\pi i\nu_t/m))$ defined by the orbifold action on the line bundles L_k at z_i acts trivially on all the line bundles occurring in the monomial W_j . In other words, the *W-spin structure* is *Broad* if there is a monomial $W_j = c_j \prod x_l^{b_{l,j}}$ in W such that for every

l with $b_{i,j} > 0$ the line bundle L_l is Broad at z_i .

Desingularization. If L is an orbifold line bundle on a smooth orbifold Riemann surface $\tilde{\Sigma}$, then the sheaf of locally invariant holomorphic sections of L is locally free of rank one, and hence dual to a unique orbifold line bundle $|L|$ on Σ . We also denote $|L|$ by ρ_*L , and it corresponds to the *desingularization* [8] of L . It can be constructed as follows.

We keep the local trivialization at other places and change it at the orbifold point z_i by a \mathbb{Z}/m -equivariant map $\Psi : (D - \{0\}) \times \mathbb{C} \rightarrow (D - \{0\}) \times \mathbb{C}$ by

$$(z, w) \rightarrow (z^m, z^{-\nu}w), \quad (7)$$

where \mathbb{Z}/m acts trivially on the second $(D - \{0\}) \times \mathbb{C}$. Then, we extend $L_{((D - \{0\}) \times \mathbb{C})}$ to a smooth holomorphic line bundle over Σ by the second trivialization. Since \mathbb{Z}/m acts trivially, this gives a line bundle over Σ , which is $|L|$. Note that if L is Broad at z_i , then $|L| = L$ locally. When L is Narrow at z_i , then $|L|$ differs from L .

Smooth metric and cylindrical metric We fix a W -spin structure

$$(L_1, \dots, L_t, \varphi_1, \dots, \varphi_s)$$

For each monomial W_i , let

$$D = - \sum_{l=1}^k \sum_{j=1}^t b_{lj} (a_j(h_l) - q_j) z_l$$

be a divisor, where $a_j(h_l)$ is the orbifold action on the line bundle L_j at the marked point z_l in the expression (6), then there is a canonical meromorphic section s_0 with divisor D . This section provides the identification

$$s_0^{-1} : K_{\Sigma} \otimes \mathcal{O}(D) \cong K_{\Sigma}(D), \quad (8)$$

where $K_{\Sigma}(D)$ is the sheaf of local, possibly meromorphic, sections of K_{Σ} with zeros determined by D . When at least one of the line bundles occurring in the monomial W_i is Narrow at z_l , then D is not effective. So the local section of $K_{\Sigma}(D)$ has zeros, and hence is a natural sub-sheaf of K_{Σ} . In general, however, it is a sub-sheaf of K_{log} . For each marked point, there is a canonical local section $\frac{dz}{z}$ of K_{log} . Using the isomorphism φ_i , there is a local section t_i of L_i with the property $W_i(t_1, \dots, t_k) = \frac{dz}{z}$. The choice of t_i is unique up to the action of the group H defined in Lemma 1.1.

We choose a metric on K_{log} with the property $|\frac{dz}{z}| = \frac{1}{|z|}$. It induces a unique metric on L_i , with property $|t_i| = |z|^{-q_i}$. Using the correspondence between L_i and $|L_i|$, it induces a metric on $|L_i|$ with the behavior $|e_i| = |z|^{a_i(h) - q_i}$ near a marked point, where e_i is the corresponding local section

of $|L_i|$. This metric on $|L_i|$ is called *smooth metric*. In particular, it is a singular metric where L is Broad ($a_i(h) = 0$) at some marked point.

If we choose a metric on K_{log} with the property $|\frac{dz}{z}| = 1$. Using the correspondence between L_i and $|L_i|$, it induces a metric on $|L_i|$ with the behavior $|e_i| = |z|^{a_i(h)}$ near a marked point, where e_i is the corresponding local section of $|L_i|$. This metric on $|L_i|$ is called *cylindrical metric*.

Let $(\Omega, z_1, \dots, z_k)$ be an obicurve with k marked points, and $B_1(z_l)$ be the unit closed disc with the center z_l . Choose a compact subset $\Omega \subset \Sigma \setminus \bigcup_{l=1}^k B_{e^{-1}}(z_l)$ such that $\{\Sigma, B_1(z_1), \dots, B_1(z_k)\}$ can cover Σ . Let $\varphi_0, \dots, \varphi_k$ be a set of partition functions subordinate to the cover. Let e_j be basis of orbifold line bundle L_j on Σ , the smooth metric is defined above: $|e_j| = |z|^{a_j(h_l) - q_j}$.

Let the section of L_j on $B_1(z_l)$ be $u_j = \tilde{u}_j e_j$, we can define norms of L^p, L_1^p as follows:

$$\begin{aligned} \|u_j\|_{p; B_1(z_l)} &= \left(\int_{B_1(z_l)} |\tilde{u}_j|^p |e_j|^p |dz d\bar{z}| \right)^{1/p} \\ \|u_j\|_{1,p; B_1(z_l)} &= \left(\int_{B_1(z_l)} (|\tilde{u}_j|^p + |\partial \tilde{u}_j|^p + |\bar{\partial} \tilde{u}_j|^p) |e_j|^p |dz d\bar{z}| \right)^{1/p} \end{aligned}$$

In the inner part of Ω which is away from the marked points, the norm is defined by the standard Sobolev norm $\|u_j\|_{W_k^p(\Omega)}$.

The global L^p, L_1^p norms is defined as :

$$\begin{aligned} \|u_j\|_p &= \|\varphi_0 u_j\|_{W_0^p(\Omega)} + \sum_{l=1}^k \|\varphi_l u_j\|_{p; B_1(z_l)}, \\ \|u_j\|_{1,p} &= \|\varphi_0 u_j\|_{W_1^p(\Omega)} + \sum_{l=1}^k \|\varphi_l u_j\|_{1,p; B_1(z_l)}, \end{aligned}$$

The weighted Sobolev space $L_1^p(\Sigma, |L_j|)$ is defined as the closure of $C_0^\infty(\Sigma \setminus (z_1, \dots, z_k), |L_j|)$ under the norm $\|\cdot\|_{1,p}$, and $L^p(\Sigma, |L_j| \otimes \wedge^{0,1})$ is defined as closure of $C_0^\infty(\Sigma \setminus (z_1, \dots, z_k), |L_j| \otimes \wedge^{0,1})$ under norm $\|\cdot\|_p$, where $\wedge^{0,1}$ is the (0,1)-form space of the Riemann surface Σ . We can define the Cauchy-Riemann operator $\bar{\partial} : L_1^p(\Sigma, |L_j|) \rightarrow L^p(\Sigma, |L_j| \otimes \wedge^{0,1})$,

For convenience, we make a coordinate transformation $z = e^{-t-i\theta}$, hence the punctured neighbourhood $B_1(z_l) \setminus \{z_l\}$ is transformed to an end $:S^1 \times [0, \infty)$.

Likewise, the operator $\bar{\partial}$ is changed as $\bar{\partial}^{t,\theta}$, and their relation is as follows:

$$\bar{\partial} = -\frac{1}{2} e^{t-i\theta} (\partial_t + i\partial_\theta) = -e^{t-i\theta} \bar{\partial}^{t,\theta}$$

The original problem is transformed as follows:

$$\bar{\partial}^{t,\theta} : \hat{W}_{1,1+k_{j,l}}^p \rightarrow W_{0,1+k_{j,l}}^p$$

where $k_{j,l} = -a_j(h_l) + q_j - 2/p$, and the spaces $\hat{W}_{1,1+k_{j,l}}^p, W_{0,1+k_{j,l}}^p$ are defined as the closure of smooth section space $\Gamma(|L_j|_{B_1(z_l)})$ under the norms $\|\cdot\|$.

$\|\cdot\|_{\hat{W}_{1,1+k_{j,l}}^p}, \|\cdot\|_{W_{0,1+k_{j,l}}^p}$ respectively:

$$\begin{aligned} \|u_j\|_{\hat{W}_{1,1+k_{j,l}}^p} &= \left(\int_{S^1 \times [0,\infty)} |\tilde{u}_j|^p e^{k_{j,l}pt} + (|\partial \tilde{u}_j|^p + |\bar{\partial} \tilde{u}_j|^p) e^{(1+k_{j,l})pt} \right)^{1/p} (9) \\ \|u_j\|_{W_{0,1+k_{j,l}}^p} &= \left(\int_{S^1 \times [0,\infty)} |\tilde{u}_j|^p e^{(1+k_{j,l})pt} \right)^{1/p} \end{aligned} \quad (10)$$

[10] has studied the index theory of $\bar{\partial}$ in case of the smooth metric, especially they got an index theorem as follows:

Theorem 1.2. ([10]) *In case of the smooth metric, if $1 < p < \frac{2}{q_j}$ and $a_j(h_l) - q_j + \frac{2}{p} \neq 1, 2$ for any $l (l = 1, \dots, k)$, then $\bar{\partial} : L_1^p(\Sigma, |L_j|) \rightarrow L^p(\Sigma, |L_j| \otimes \wedge^{0,1})$ is a Fredholm operator.*

In particular, if $2 < p < \frac{2}{1-\delta_j}$ we have the index relation

$$\begin{aligned} &\text{ind}(\bar{\partial} : L_1^p(\Sigma, |L_j|) \rightarrow L^p(\Sigma, |L_j| \otimes \wedge^{0,1})) \\ &= \text{ind}(\bar{\partial}^{t,\theta} : W_{1,1+\kappa}^p \rightarrow W_{0,1+\kappa}^p) + \#\{z_l : c_{jl} < 0\} \end{aligned}$$

and the index is independent of $p \in (2, \frac{2}{1-\delta_j})$. Where $\bar{\delta}_j = \min_{l: c_{jl} > 0} (c_{jl})$, $c_{jl} = a_j(h_l) - q_j$.

For convenience, we call this theorem as the index transformation theorem. Using this theorem, concrete index computation of $\bar{\partial}$ is transformed to compute $\text{ind}(\bar{\partial}^{t,\theta} : W_{1,1+\kappa}^p \rightarrow W_{0,1+\kappa}^p)$.

Now we can state our first main theorem as follows:

Theorem 1.3. *In case of the smooth metric, we can compute the index of $\bar{\partial}$ as follows:*

$$\begin{aligned} &\text{ind}(\bar{\partial} : L_1^p(\Sigma, |L_j|) \rightarrow L^p(\Sigma, |L_j| \otimes \wedge^{0,1})) \\ &= k + (1 - 2q_j)(2 - 2g - k) - 2 \sum_{l=1}^k a_j(h_l) + \#\{z_l : c_{jl} < 0\} \end{aligned}$$

where $c_{jl} = a_j(h_l) - q_j$, q_j is the fractional degree of a nondegenerate quasi-homogeneous polynomial W with respect to the j th variable, $a_j(h_l) = v_{j,l}/m_l$ is the orbifold action on the line bundle L_j at the marked point z_l .

And the right side is actually an integer, i.e.

$$k + (1 - 2q_j)(2 - 2g - k) - 2 \sum_{l=1}^k a_j(h_l) + \#\{z_l : c_{jl} < 0\} \in \mathbb{Z}$$

As for the cylindrical metric, we can similarly define the operator

$$\bar{\partial} : L_1^p(\Sigma, |L_j|) \rightarrow L^p(\Sigma, |L_j| \otimes \wedge^{0,1})$$

only by replacing the smooth metric $|e_j| = |z|^{a_j(h_l) - q_j}$ by the cylindrical metric $|e_j| = |z|^{a_j(h_l)}$.

In [12], the authors introduced the linearized operator D of the Witten map, that is:

$$D = D_{\varphi, \mu} WI : L_1^p(\Sigma, L_1 \times L_2 \times \cdots \times L_N) \rightarrow L^p(\Sigma, L_1 \otimes \wedge^{0,1}) \times \cdots \times L^p(\Sigma, L_N \otimes \wedge^{0,1})$$

Then they proved this operator is a Fredholm operator under some mild conditions in case of the cylindrical metric and computed its index. If we add weight δ to Sobolev spaces above, and consider the following operator:

$$D^\delta = D_{\varphi, \mu} WI : L_1^{p, \delta}(\Sigma, L_1 \times L_2 \times \cdots \times L_N) \rightarrow L^{p, \delta}(\Sigma, L_1 \otimes \wedge^{0,1}) \times \cdots \times L^{p, \delta}(\Sigma, L_N \otimes \wedge^{0,1})$$

we can ask what is the relation between $\text{index}(D^\delta)$ and δ ?

Our second main theorem can totally solve this problem. Let's state it as follows:

Theorem 1.4. *Assuming that $D^\delta, D^{\delta'}$ stand for the linearized operator D of the Witten map with weights δ, δ' respectively. In case of the cylindric metric, we have the index jumping formula*

$$\text{ind}(D^\delta) - \text{ind}(D^{\delta'}) = \sum_{j=1}^N \sum_{l=1}^k ([\delta'_{j,l}] - [\delta_{j,l}]).$$

where $\delta = (\delta_{jl}) \in \mathbb{R}^{N \times k}, \delta' = (\delta'_{jl}) \in \mathbb{R}^{N \times k}$ are weight matrixes, N stands for the number of variable in a nondegenerate quasi-homogeneous polynomial W , k stands for the number of marked points.

The paper is organized as follows. In section 2, we will first review Riemann-Roch theorem with boundary, Donaldson index theory and Lockhat-McOwen theory, then as an application of their work we prove Theorem 1.3. In section 3, we will prove Theorem 1.4 by generalizing Theorem 1.2.

2 Index computation in case of the smooth metric

In this section, we will prove Theorem 1.3, which is equivalent to compute

$$\text{ind}(\bar{\partial} : L_1^p(\Sigma, |L_j|) \rightarrow L^p(\Sigma, |L_j| \otimes \wedge^{0,1})) \quad (11)$$

in case of the smooth metric. By Theorem 1.2, this index problem can be transformed to compute

$$\text{ind}(\bar{\partial}^{t, \theta} : W_{1, 1+\kappa}^p \rightarrow W_{0, 1+\kappa}^p) \quad (12)$$

Let e_j be a basis of orbifold line bundle L_j on Σ , and recall that the smooth metric is defined: $|e_j| = |z|^{a_j(h_l) - q_j}$, where q_j is the fractional degree of a nondegenerate quasi-homogeneous polynomial W with respect to the j th variable, $a_j(h_l)$ is the orbifold action on the line bundle L_j at the marked point z_l .

Note that q_j is a nonnegative rational number, and $a_j(h_l) = 0$ when L_j is Broad (see section 1) at z_l . So $a_j(h_l) - q_j < 0$ may happen. In that case the marked point z_l is a singularity with respect to the smooth metric. Therefore, the smooth metric is a singular metric and we can not directly compute the index of the operator $\bar{\partial} : L_1^p(\Sigma, |L_j|) \rightarrow L^p(\Sigma, |L_j| \otimes \wedge^{0,1})$.

Let z_l be a marked point, and consider the restriction of the bundle $|L_j|_{B_1(z_l)}$ on the disc $B_1(z_l)$. Assume that $B_1(z_l) \times \mathbb{C} \rightarrow |L_j|_{B_1(z_l)} : (z, w) \rightarrow \Psi_l(z)w$ is a trivialization such that $\Psi_l(e^{i\theta})\mathbb{R}$ forms a totally real bundle on $S_l^1 = \partial B_1(z_l)$. Define spaces

$$\begin{aligned}\hat{W}_{1,1+\kappa_{j,l}}^{p,B} &:= \{\tilde{u}_j \in \hat{W}_{1,1+\kappa_{j,l}}^p \mid \text{and } \tilde{u}_j(e^{i\theta}) \in \Psi_l(e^{i\theta})\mathbb{R}\} \\ W_{1,1+\kappa_{j,l}}^{p,B} &:= \{\tilde{u}_j \in W_{1,1+\kappa_{j,l}}^p \mid \text{and } \tilde{u}_j(e^{i\theta}) \in \Psi_l(e^{i\theta})\mathbb{R}\}\end{aligned}$$

where $\hat{W}_{1,1+\kappa_{j,l}}^p, W_{1,1+\kappa_{j,l}}^p$ are defined as (9),(10), $\kappa_{j,l} = -a_j(h_l) + q_j - 2/p$.

We can also define the space

$$W_1^{p,B}(\text{inn}) := \{\tilde{u} \in W_1^p(\Sigma \setminus \cup_l B_1(z_l)) \mid \tilde{u}(e^{i\theta}) \in \Psi_l(e^{i\theta})\mathbb{R} \text{ for } e^{i\theta} \in S_l^1\}$$

where S_l^1 is the boundary of $B_1(z_l)$.

Now consider local index problems $\text{ind}(\bar{\partial}^{t,\theta} : W_1^{p,B}(\text{inn}) \rightarrow W_0^p(\text{inn}))$, $\text{ind}(\bar{\partial}^{t,\theta} : \hat{W}_{1,1+\kappa_{j,l}}^{p,B} \rightarrow W_{0,1+\kappa_{j,l}}^p), \text{ind}(\bar{\partial}^{t,\theta} : W_{1,1+\kappa_{j,l}}^{p,B} \rightarrow W_{0,1+\kappa_{j,l}}^p), l = 1, \dots, k$. [10] has got index decomposition theorems which relate local index problems above to $\text{ind}(\bar{\partial}^{t,\theta} : \hat{W}_{1,1+\kappa}^p \rightarrow W_{0,1+\kappa}^p)$, $\text{ind}(\bar{\partial}^{t,\theta} : W_{1,1+\kappa}^{p,B} \rightarrow W_{0,1+\kappa}^p)$ as follows:

Theorem 2.1. ([10]) *In case of the smooth metric, we have*

$$\begin{aligned}1) & \text{ind}(\bar{\partial}^{t,\theta} : \hat{W}_{1,1+\kappa}^p \rightarrow W_{0,1+\kappa}^p) \\ &= \text{ind}(\bar{\partial}^{t,\theta} : W_1^{p,B}(\text{inn}) \rightarrow W_0^p(\text{inn})) + \sum_{l=1}^k \text{ind}(\bar{\partial}^{t,\theta} : \hat{W}_{1,1+\kappa_{j,l}}^{p,B} \rightarrow W_{0,1+\kappa_{j,l}}^p) \\ 2) & \text{ind}(\bar{\partial}^{t,\theta} : W_{1,1+\kappa}^p \rightarrow W_{0,1+\kappa}^p) \\ &= \text{ind}(\bar{\partial}^{t,\theta} : W_1^{p,B}(\text{inn}) \rightarrow W_0^p(\text{inn})) + \sum_{l=1}^k \text{ind}(\bar{\partial}^{t,\theta} : W_{1,1+\kappa_{j,l}}^{p,B} \rightarrow W_{0,1+\kappa_{j,l}}^p)\end{aligned}$$

This index decomposition theorem is our start of later index computation. By this theorem, to compute $\text{ind}(\bar{\partial}^{t,\theta} : W_{1,1+\kappa}^p \rightarrow W_{0,1+\kappa}^p)$ is equivalent to compute

$$\text{ind}(\bar{\partial}^{t,\theta} : W_1^{p,B}(\text{inn}) \rightarrow W_0^p(\text{inn}))$$

and

$$\text{ind}(W_{1,1+\kappa_{j,l}}^{p,B} \rightarrow W_{0,1+\kappa_{j,l}}^p)$$

Remark 2.1. This theorem is in case of the smooth metric. In section 3, we will generalize this theorem in case of the cylindrical metric.

2.1 Riemann-Roch theorem with boundary

First let's focus on the computation problem of

$$\text{ind}(\bar{\partial}^{t,\theta} : W_1^{p,B}(\text{inn}) \rightarrow W_0^p(\text{inn}))$$

McDuff and Salamon ([14]) have already studied similar problems and got the Riemann-Roch theorem with boundary. As an application of their work, we will compute

$$\text{ind}(\bar{\partial}^{t,\theta} : W_1^{p,B}(\text{inn}) \rightarrow W_0^p(\text{inn}))$$

Note that orbifold structure of L_j near marked point z_l is given by:

$$e^{\frac{2\pi i}{m}}(z, m) = (e^{\frac{2\pi i}{m}}z, e^{2\pi i(a_j(h_l))}w) = (e^{\frac{2\pi i}{m}}z, e^{2\pi i(\frac{v_{j,l}}{m})}w) \quad (13)$$

According to the definition of L_j , there is a natural boundary condition for $|L_j|$, that is:

$$\Psi_l(e^{i\theta})\mathbb{R} = e^{iv_{j,l}\theta}\mathbb{R} \quad (14)$$

Remark 2.2. Note that we delete the factor 2π in above equality for complying with the framework of the appendix C in [14], but it will not influence our later computation.

Now we can apply [14]'s Riemann-Roch theorem with boundary to our problem. We cite their theorem (Theorem C.1.10 of [14]) as follows:

Theorem 2.2 (Riemann-Roch theorem with boundary). *Let $E \rightarrow \Sigma$ be a complex vector bundle on a compact Riemannian surface with boundary and $F \subset E|_{\partial\Sigma}$ be a real subbundle. Let D be a real Cauchy–Riemann operator on E of class $W^{l-1,p}$, where l is a positive integer and $p > 1$ such that $lp > 2$. Then the following holds for every integer $k \in \{1, 2, \dots, l\}$ and every real number $q > 1$ such that $k - \frac{2}{q} \leq l - \frac{2}{p}$.*

(1) The operators

$$\begin{aligned} D_F &: W_F^{k,q}(\Sigma, E) \rightarrow W^{k-1,q}(\Sigma, \wedge^{0,1}T^*\Sigma \otimes E), \\ D_F^* &: W_F^{k,q}(\Sigma, \wedge^{0,1}T^*\Sigma \otimes E) \rightarrow W^{k-1,q}(\Sigma, E) \end{aligned}$$

are Fredholm operators.

(2) The real Fredholm index of D_F is given by

$$\text{ind}(D_F) = n\chi(\Sigma) + \mu(E, F) \quad (15)$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ , $\mu(E, F)$ is the boundary Maslov index (see the Appendix), n is the complex rank of E .

Note that $\Sigma \setminus \cup_{l=1}^k B_1(z_l)$ is a Riemannian surface of genus g with boundary, $E = |L_j|_{\Sigma \setminus \cup_{l=1}^k B_1(z_l)}$ is a complex vector bundle on $\Sigma \setminus \cup_{l=1}^k B_1(z_l)$, $F = |L_j|_{\cup_{l=1}^k \partial(B_1(z_l))}$ is a real subbundle of E , and $\partial^{t,\theta}$ is a real Cauchy-Riemann operator on E . By Riemann-Roch theorem with boundary above, we get

$$\text{ind}(W_1^{p,B}(\text{inn}) \rightarrow W_0^p(\text{inn})) = \chi(\Sigma) + \mu(E, F) = (2 - 2g - k) + \mu(E, F) \quad (16)$$

Therefore, it suffices to compute the boundary Maslov index $\mu(E, F)$.

Let $\Sigma_{01} = \Sigma \setminus \cup_{l=1}^k B_1(z_l)$, Σ_{12} disjoint union of k closed unit discs, $E_{01} = E, F_{01} = F$, E_{12}, F_{12} bundles on Σ_{12} with the same boundary conditions as E, F respectively, Σ_{02} closed Riemannian surface corresponding to Σ_{01} , E_{02} extension of E on Σ_{02} , $F_{02} = \emptyset$.

According to the definition of decomposition (see Definition 4.1, 4.2 in the Appendix), it is easy to check that $(\Sigma_{01}, \Sigma_{12})$ is a decomposition of Σ_{02} , $(E_{01}, F_{01}), (E_{12}, F_{12})$ are a bundle pair decomposition of (E_{02}, F_{02}) . So, by the composition axiom in Theorem 4.1, we have:

$$\mu(E, F) = \mu(E_{02}, \emptyset) - \mu(E_{12}, F_{12}) \quad (17)$$

By Theorem 4.2,

$$\mu(E_{02}, \emptyset) = 2c_1(E_{02})[\Sigma_{02}] = 2c_1(|L_j|)([\Sigma_{02}]) = 2\deg(|L_j|) \quad (18)$$

Therefore the original index problem can be translated to be a degree computation of the line bundle $|L_j|$. This has been done in [11].

Theorem 2.3. ([11]) We can compute the degree of $|L_j|$ as follows

$$\deg(|L_j|) = q_j(2g - 2 + k) - \sum_{l=1}^k a_j(h_l) \in \mathbb{Z} \quad (19)$$

Therefore, by (18), (19) we get

$$\mu(E_{02}, \emptyset) = 2q_j(2g - 2 + k) - 2 \sum_{l=1}^k a_j(h_l) \in \mathbb{Z} \quad (20)$$

The remaining problem is to compute the Maslov index $\mu(E_{12}, F_{12})$. It is easy considering our given boundary conditions (14), by Theorem 4.1, we get

$$\mu(E_{12}, F_{12}) = -2 \sum_{l=1}^k v_{j,l} \quad (21)$$

By (17), (20), (21), we get

$$\mu(E, F) = 2q_j(2g - 2 + k) - 2 \sum_{l=1}^k a_j(h_l) + 2 \sum_{l=1}^k v_{j,l} \quad (22)$$

Combining Theorem 2.3, (16) and (22), we get

Theorem 2.4.

$$\begin{aligned} & \text{ind}(\bar{\partial}^{t,\theta} : W_1^{p,B}(\text{inn}) \rightarrow W_0^p(\text{inn})) \\ &= (1 - 2q_j)(2 - 2g - k) - 2 \sum_{l=1}^k a_j(h_l) + 2 \sum_{l=1}^k v_{j,l} \in \mathbb{Z} \end{aligned}$$

2.2 L^p -index gluing theorem

Now let us compute the index

$$\text{ind}(\bar{\partial}^{t,\theta} : W_{1,1+\kappa_{j,l}}^{p,B} \rightarrow W_{0,1+\kappa_{j,l}}^p)$$

Note that here the base manifold is an end $S^1 \times [0, \infty)$, so this is an index problem on noncompact manifold on which classical Atiyah-Singer theorems ([1-7]) do not work. However, this kind of manifold is the easiest case of noncompact manifold, whose index theory has been studied by Donaldson, Lockhart, McOwen ([9],[13]), etc. We need apply and generalize their results to solve our problems.

In [9], Donaldson studied the index theory over a tubular manifold and got index gluing theorems (see the Appendix). In fact, he mainly considered the 4-dimension case, but his theorems also can fit for general case. Here as an application of his work, we will prove the L^p -index gluing theorem with weights for 2-dimension case.

First consider the following gluing problem with only two ends. Assume that Riemannian surface Σ is a disjoint union of two disconnected components $\Sigma = \Sigma_1 \cup \Sigma_2$, and two ends $Y \times (0, \infty), \bar{Y} \times (0, \infty)$ whose orientations are opposite are contained in different components, where $Y = S^1$. Suppose there are vector bundles E_1, E_2 on Σ_1, Σ_2 respectively. Then we can define Sobolev spaces $L^p(E_i), L_1^p(E_i)$ and differential operators

$$D_i : L_1^p(E_i) \rightarrow L^p(E_i)$$

Assume that D_i can be written as $D_i = \frac{d}{dt} + L_i, i = 1, 2$, where L_i are self-dual elliptic operators.

Now we consider the Riemannian surfaces Σ^\sharp obtained by identifying the two ends of Σ , then construct E^\sharp over Σ^\sharp and the operator $D^\sharp : L_1^p(E^\sharp) \rightarrow L^p(E^\sharp)$ (see [9] or the Appendix for more details).

We can also prove these operators are Fredholm operators, then define their Fredholm indices $\text{ind}(D_i), \text{ind}(D^\sharp)$ as [9]. Moreover we can prove

Theorem 2.5 (L^p -Index gluing theorem). *In situations above, assume that operators L_1, L_2 are invertible in the decomposition $D_1 = \frac{d}{dt} + L_1, D_2 = \frac{d}{dt} + L_2$, we have*

$$\text{ind}(D^\sharp) = \text{ind}(D_1) + \text{ind}(D_2) \quad (23)$$

Proof. See the Appendix. \square

Remark 2.3. The key point of the proof is that differential operators D_i can be decomposed as $D_i = \frac{d}{dt} + L_i$ as Donaldson did. Therefore, it is similar to his proof of L^2 -edition.

When the operators $L_i (i = 1, 2)$ are not invertible, we must introduce weights $\alpha_i \in \mathbb{R}$ and consider weighted Sobolev spaces $L_1^{p, \alpha_i}(E_i), L^{p, \alpha_i}(E_i)$, and

$$D^{\alpha_i} : L_1^{p, \alpha_i}(E_i) \rightarrow L^{p, \alpha_i}(E_i), i = 1, 2$$

When $\alpha_2 = -\alpha_1 = -\alpha$, we can similarly glue the two ends and get

$$D^\sharp : L_1^{p, \alpha}(E^\sharp) \rightarrow L^{p, \alpha}(E^\sharp)$$

However, the introduction of weighted Sobolev space is equivalent to replace the operator L of $(L - \alpha)$ in Sobolev space without weights. Therefore, we can easily generalize the index gluing theorem above to the case with weights as follows (see [9]).

Theorem 2.6 (L^p -Index gluing theorem with weights I). *Assuming $\alpha \in \mathbb{R}$ such that $L_i - \alpha (i = 1, 2)$ is invertible, we have*

$$\text{ind}(D^\sharp) = \text{ind}(D^\alpha) + \text{ind}(D^{-\alpha}) \quad (24)$$

We can further consider weight vector case which corresponds to more ends. Choose a weight α_i for each end $Y_i \times (0, \infty)$ of Σ and define a weight vector $\vec{\alpha} = (\alpha_1, \dots, \alpha_N)$. Fix a positive function W on Σ which is equal to $e^{\alpha_i t}$ on the i th end and define norms

$$\|f\|_{L^{p, \vec{\alpha}}} = \|Wf\|_{L^p}, \|f\|_{L_1^{p, \vec{\alpha}}} = \|Wf\|_{L_1^p}$$

then we can define $L^{p, \vec{\alpha}}, L_1^{p, \vec{\alpha}}$ and

$$D^{\vec{\alpha}} : L_1^{p, \vec{\alpha}} \rightarrow L^{p, \vec{\alpha}}$$

Similar to [9], we can easily obtain the index gluing theorem in weight vector case.

Theorem 2.7 (L^p -Index gluing theorem with weights II).

$$\text{ind}(D^{\sharp;(\alpha_2, \dots, \alpha_N)}) = \text{ind}(D^{(\alpha_1, \alpha_2, \dots, \alpha_N)}) + \text{ind}(D^{(-\alpha_1, \alpha_2, \dots, \alpha_N)}) \quad (25)$$

2.3 Lockhart-McOwen theory

In this part, we recollect the work of Lockhart and McOwen ([13]) for general elliptic operators defined on a noncompact manifolds with finite ends. In next section we will compute the index using their work.

Suppose X is an n -dimensional noncompact manifold without boundary, containing a compact set X_0 such that

$$X \setminus X_0 = \{(\omega, \tau) : \omega \in \Omega, \tau \in (0, \infty)\}$$

where Ω is a $n - 1$ -dimensional closed Riemannian manifold with a smooth measure $d\omega$.

Let E, F be rank- d vector bundles over X . Denote by $C^\infty(E)$ the set of smooth sections and $C_0^\infty(E)$ the set of smooth sections with compact supported sets. Choose a finite cover $\{\Omega_1, \dots, \Omega_N\}$ of coordinate patches of Ω and let $X_v = \Omega_v \times (0, +\infty)$. We can continue to choose a covering X_{N+1}, \dots, X_M of coordinate patches of X_0 such that E can be trivialized over $X_v, v = 1, \dots, N, \dots, M$. Let $u = (u_1, \dots, u_d)$ be a trivialization of a section u with compact supported set over X_v , we can define the norm

$$\|u\|_{W_s^p(X_v)} := \sum_{|\alpha| \leq s} \sum_{l=1}^d \|D^\alpha u_l\|_{W_0^p(X_v)}, (D = -i\partial/\partial x)$$

where we use the measure $d\omega d\tau$ if $v = 1, \dots, N$. Let $\varphi_1, \dots, \varphi_{N+M}$ be a set of C^∞ partition functions subordinate to the cover X_1, \dots, X_{N+M} . We define a norm on $C_0^\infty(E)$ by

$$\|u\|_{W_s^p} := \sum_{v=1}^{N+M} \|\varphi_v u\|_{W_s^p(X_v)}$$

and let $W_s^p(E)$ be the closure of $C_0^\infty(E)$ in this norm. We can add a weight at infinity to generalize this space. Over $X_v, v = 1, \dots, N$ we define the weighted norm

$$\|u\|_{W_{s,k}^p(X_v)} := \sum_{|\alpha| \leq s} \sum_{v=1}^d \|e^{k\tau} D^\alpha u_l\|_{W_0^p(X_v)}$$

and replace $W_s^p(E)$ by $W_{s,k}^p(E)$ whose norm is given below

$$\|u\|_{W_{s,k}^p} := \sum_{v=1}^N \|\varphi_v u\|_{W_{s,k}^p(X_v)} + \sum_{v=N+1}^{N+M} \|\varphi_v u\|_{W_s^p(X_v)}$$

Similarly we can define $W_{r,k}^p(F)$, where $s = (s_1, \dots, s_I), r = (r_1, \dots, r_J)$ are multiple indices. Suppose $A : C_0^\infty(E) \rightarrow C_0^\infty(F)$ is translation invariant elliptic operator with respect to (s, r) . Then $A : W_{s,k}^p(E) \rightarrow W_{r,k}^p(F)$ is a bounded operator. Furthermore, Lockhart and McOwen([13]) proved the following theorem

Theorem 2.8 (Index jumping formula). *Suppose A is elliptic with respect to (r, s) and is translation invariant when $\tau > 0$. Then we have:*

(1) *There exists a discrete subset $\mathfrak{D}_A \subset \mathbb{R}$ such that the operator:*

$A : W_{s,k}^p(E) \rightarrow W_{r,k}^p(F)$ is Fredholm operator if and only if $k \in \mathbb{R} \setminus \mathfrak{D}_A$.

(2) *For $k_1, k_2 \in \mathbb{R} \setminus \mathfrak{D}_A$ with $k_1 < k_2$, there is*

$$i_{k_1}(A) - i_{k_2}(A) = N(k_1, k_2) \quad (26)$$

where i_{k_j} is the Fredholm index of $A : W_{s,k_j}^p(E) \rightarrow W_{r,k_j}^p(F), j = 1, 2$,

$$N(k_1, k_2) = \Sigma\{d(\lambda) : \lambda \in \mathfrak{E}_A, k_1 < \text{Im}(\lambda) < k_2\} \quad (27)$$

where \mathfrak{E}_A is the spectrum of A , $\mathfrak{D}_A := \{\text{Im}(\lambda) \in \mathbb{R} : \lambda \in \mathfrak{E}_A\}$, $d(\lambda)$ is the dimension of the eigenspace corresponding to the spectrum point λ .

2.4 Index computation

Proof of Theorem 1.3. First we compute

$$\text{ind}(\bar{\partial}^{t,\theta} : W_{1,1+\kappa_{j,l}}^{p,B} \rightarrow W_{0,1+\kappa_{j,l}}^p)$$

where $\kappa_{j,l} = -a_j(h_l) + q_j - 2/p$.

We have a decomposition $\bar{\partial} = \partial_t + i\partial_\theta = \frac{d}{dt} + L$. Note that $L = i\partial_\theta$ is not invertible (Because its spectrum is \mathbb{Z} , and $\ker L \cong \mathbb{C}$), so we need introduce weights as discussion above .

Naturally consider the number $1 + \kappa_{j,l}$ as a weight: $\alpha = \alpha^+ = 1 + \kappa_{j,l}$, and define $\alpha^- = -\alpha^+$. Write $\bar{\partial}^{\alpha^+}, \bar{\partial}^{\alpha^-}$ as $\bar{\partial}^+, \bar{\partial}^-$ respectively. Then we have $\text{ind}(\bar{\partial}) = \text{ind}(\bar{\partial}^+)$. By L^p -index gluing theorem with weights I (Theorem 2.6), we get

$$\text{ind}(\bar{\partial}^\sharp) = \text{ind}(\bar{\partial}^+) + \text{ind}(\bar{\partial}^-) \quad (28)$$

where the operator $\bar{\partial}^\sharp$ is defined over the compact Riemann surface $S^1 \times I$, the boundary conditions only need reverse (14),

$$\Psi_l(e^{i\theta})\mathbb{R} = e^{-iv_{j,l}\theta}\mathbb{R} \quad (29)$$

By Theorem 2.2, we get :

$$\text{ind}(\bar{\partial}^\sharp) = \chi(S^1 \times I) + \mu(E, F) = \mu(E, F) \quad (30)$$

Therefore, the computation of $\text{ind}(\bar{\partial}^\sharp)$ can be transformed to be computation of boundary Maslov index $\mu(E, F)$.

We adopt previous methods. First consider a disjoint union of two discs with opposite boundary conditions as $S^1 \times I$, then glue the two discs on $S^1 \times I$ along the boundary. So by Theorem 2.2, Theorem 4.1 and Theorem 4.2, we get :

$$\text{ind}(\bar{\partial}^\sharp) = \mu(E, F) = 4 - 4v_{jl} \quad (31)$$

On the other hand, by our index jumping formula (Theorem 2.8) we get

$$\text{ind}(\bar{\partial}^-) - \text{ind}(\bar{\partial}^+) = \dim_{\mathbb{R}} \mathbb{C} = 2. \quad (32)$$

Therefore by (28), (31), (32), we get

Theorem 2.9.

$$\text{ind}(\bar{\partial}^{t,\theta} : W_{1,1+\kappa_{j,l}}^{p,B} \rightarrow W_{0,1+\kappa_{j,l}}^p) = 1 - 2v_{jl} \quad (33)$$

Combining Theorem 1.2, Theorem 2.1, Theorem 2.4, and Theorem 2.9, we ultimately get

$$\begin{aligned} & \text{ind}(\bar{\partial} : L_1^p(\Sigma, |L_j|) \rightarrow L^p(\Sigma, |L_j| \otimes \wedge^{0,1})) \\ &= k + (1 - 2q_j)(2 - 2g - k) - 2 \sum_{l=1}^k a_j(h_l) + \sharp\{z_l : c_{jl} < 0\} \end{aligned}$$

where $c_{jl} = a_j(h_l) - q_j$, q_j is the fractional degree of a nondegenerate quasi-homogeneous polynomial W with respect to the j th variable, $a_j(h_l) = v_{j,l}/m_l$ is the orbifold action on the line bundle L_j at the marked point z_l .

And the right side is actually an integer, i.e.

$$k + (1 - 2q_j)(2 - 2g - k) - 2 \sum_{l=1}^k a_j(h_l) + \sharp\{z_l : c_{jl} < 0\} \in \mathbb{Z}$$

This completes the proof of Theorem 1.3. \square

Remark 2.4. If we write index formula above as

$$\begin{aligned} & \text{ind}(\bar{\partial} : L_1^p(\Sigma, |L_j|) \rightarrow L^p(\Sigma, |L_j| \otimes \wedge^{0,1})) \\ &= 2(1 - 2q_j)(1 - g) - 2 \sum_{l=1}^k (\Theta_j^{\gamma_l} - q_j) + \sharp\{z_l : c_{jl} < 0\} \end{aligned}$$

where $\Theta_j^{\gamma_l} = a_j(h_l)$. We can see that this index formula is almost the same as the index formula ([12]) in case of the cylindrical metric except for the term $\sharp\{z_l : c_{jl} < 0\}$.

3 Index computation in case of the cylindrical metric

In [12], they introduced the linearized operator D of Witten map, that is:

$$D = D_{\varphi, \mu} WI : L_1^p(\Sigma, L_1 \times L_2 \times \cdots \times L_N) \rightarrow L^p(\Sigma, L_1 \otimes \wedge^{0,1}) \times \cdots \times L^p(\Sigma, L_N \otimes \wedge^{0,1})$$

Then they proved this operator is a Fredholm operator under some mild conditions in case of the cylindrical metric and computed its index. The next interesting question is: If we add weight δ to Sobolev spaces above, and consider the following operator:

$$D^\delta = D_{\varphi, \mu} WI : L_1^{p, \delta}(\Sigma, L_1 \times L_2 \times \cdots \times L_N) \rightarrow L^{p, \delta}(\Sigma, L_1 \otimes \wedge^{0,1}) \times \cdots \times L^{p, \delta}(\Sigma, L_N \otimes \wedge^{0,1})$$

then what is the relation between $\text{ind}(D^\delta)$ and δ ?

For convenience, let's first consider the operator $\bar{\partial}_j^\delta$ defined in weighted Sobolev space

$$\bar{\partial}_j^\delta : L_1^{p, \delta}(\Sigma, |L_j|) \rightarrow L^{p, \delta}(\Sigma, |L_j| \otimes \wedge^{0,1}), j = 1, \dots, N$$

where $|L_j|$ is the desingularization of orbifold line bundle L_j .

Let $(\Sigma, z_1, \dots, z_k)$ be an orbicurve with k marked points, $B_1(z_l)$ unit closed disc with the center z_l . Choose a compact subset $\Omega \subset \Sigma \setminus \cup B_{\epsilon^{-1}}(z_l)$ such that $\Sigma, B_1(z_1), \dots, B_1(z_k)$ can cover Σ . Let $\varphi_0, \dots, \varphi_k$ be a set of partition functions subordinate to the cover. Let e_j be basis of orbifold line bundle L_j on Σ , and recall the cylindrical metric is defined as: $|e_j| = |z|^{a_j(z_l)}$. Let section of L_j on $B_1(z_1)$ be $u_j = \tilde{u}_j e_j$, we can define norm $\|\cdot\|_p, \|\cdot\|_{1,p}$, $L_1^p(\Sigma, |L_j|)$, and the operator $\bar{\partial} : L_1^p(\Sigma, |L_j|) \rightarrow L^p(\Sigma, |L_j| \otimes \wedge^{0,1})$ almost the same as the smooth metric case (see section 1).

Likewise, we also make a coordinate transformation $z = e^{-t-i\theta}$ and change $\bar{\partial}$ as $\bar{\partial}^{t, \theta}$, and their relation is as follows:

$$\bar{\partial} = -\frac{1}{2}e^{t-i\theta}(\partial_t + i\partial_\theta) = -e^{t-i\theta}\bar{\partial}^{t, \theta}$$

We write $\bar{\partial}_j^\delta$ as $\bar{\partial}^\delta$ if no confusion occurs. So

$$\bar{\partial}^\delta : L_1^{p, \delta}(\Sigma, |L_j|) \rightarrow L^{p, \delta}(\Sigma, |L_j| \otimes \wedge^{0,1})$$

is transformed as follows:

$$\bar{\partial}^{t, \theta, \delta} : \hat{W}_{1, 1+k+\delta}^p \rightarrow W_{0, 1+k+\delta}^p$$

where norms $\hat{W}_{1, 1+k+\delta}^p, W_{0, 1+k+\delta}^p$ are defined as :

$$\begin{aligned} \|u_j\|_{\hat{W}_{1, 1+k+\delta}^p} &= \left(\int_{S^1 \times [0, \infty)} |\tilde{u}_j|^p e^{(k+\delta)pt} + (|\partial \tilde{u}_j|^p + |\bar{\partial} \tilde{u}_j|^p) e^{(1+k+\delta)pt} \right)^{1/p} \\ \|u_j\|_{W_{0, 1+k+\delta}^p} &= \left(\int_{S^1 \times [0, \infty)} |\tilde{u}_j|^p e^{(1+k+\delta)pt} \right)^{1/p} \end{aligned}$$

3.1 Index transformation theorem

Unfortunately, the space $\hat{W}_{1,1+k+\delta}^p$ is not a normal weighted Sobolev space (The normal one is $W_{1,1+k+\delta}^p$), so we can not directly apply Lockhart-McOwen theory above. Therefore, first we should transform this problem into a normal case, which needs generalize the index transformation theorem (Theorem 1.2) to the case of the cylindrical metric. That is, we want to prove

Theorem 3.1. (*Index transformation theorem*) *In case of the cylindrical metric, if $1 < p < \frac{2}{q_j}$ and $a_j(h_l) + \frac{2}{p} \neq 1, 2$ for any $l (l = 1, \dots, k)$, then $\bar{\partial} : L_1^p(\Sigma, |L_j|) \rightarrow L^p(\Sigma, |L_j| \otimes \wedge^{0,1})$ is a Fredholm operator.*

In particular, if $p > 2$ we have the index relation

$$\begin{aligned} & \text{ind}(\bar{\partial} : L_1^p(\Sigma, |L_j|) \rightarrow L^p(\Sigma, |L_j| \otimes \wedge^{0,1})) \\ &= \text{ind}(\bar{\partial}^{t,\theta} : W_{1,1+\kappa}^p \rightarrow W_{0,1+\kappa}^p) + \#\{z_l : c_{jl} = 0\} \end{aligned}$$

and the index is independent of p in the interval $(2, \infty)$.

Next we will prove this theorem step by step as [10].

Remark 3.1. The parameter c_{ij} is very important, where $c_{jl} = a_j(h_l) - q_j$ in case of the smooth metric, and $c_{jl} = a_j(h_l)$ in case of the cylindrical metric. It is obvious that both cases $c_{jl} \geq 0, c_{jl} < 0$ can happen in case of the smooth metric, but only the case $c_{jl} \geq 0$ can happen in case of the cylindrical metric.

We still use the same notations as [10]. Firstly we can obtain local estimate of special solution in case of the cylindrical metric similar to Lemma 4.3 in [10]:

Lemma 3.2. *If $f \in L^p(B_1(0), |L_j| \otimes \wedge^{0,1})$ for p satisfying the condition $a_{j,l} = a_j(h_l) + 2/p \in \mathbb{R} \setminus \mathbb{Z}$, then the special solution $u_s = Q_s \circ f$ satisfies the following estimates:*

(1) *if $1 < p < \infty$, then*

$$\|u_s\|_{1,p;B_1(0)} + \left\| \frac{u_s}{z} \right\|_{p;B_1(0)} \leq C \|f\|_{p;B_1(0)} \quad (34)$$

(2) *if $1 < p \leq 2$, and $1 < q < \frac{2p}{2-p}$, then*

$$\|u_s\|_{q;B_1(0)} \leq C \|u_s\|_{1,p;B_1(0)} \leq C \|f\|_{p;B_1(0)}; \quad (35)$$

(3) *if $p > 2$, and $0 < \alpha < 1 - \frac{2}{p}$, then*

$$\|\tilde{u}_s r^c\|_{C^\alpha(B_1(0))} \leq C \|u_s\|_{1,p;B_1(0)} \leq C \|f\|_{p;B_1(0)} \quad (36)$$

where $c = a_j(h_l)$.

Proof. Through serious check on Lemma 4.3 in [10], we can find the whole proof views the constant $c = a_j(h_l) - q_j$ as a unity. So the proof does not change if we replace $c = a_j(h_l) - q_j$ as $c = a_j(h_l)$ in case of cylindrical metric. \square

Next we consider estimate of the homogeneous solution in case of the cylindrical metric. Compared to estimate of the homogeneous solution in case of the smooth metric, we may find it is easier to be dealt with because there is only one case (see Remark 3.1). We have the following lemma similar to Lemma 4.4 in [10].

Lemma 3.3. *Let $\bar{\partial}u = 0$ and $u \in L^p(B_1^+(0), |L_j|)$ for $p > 1$. We have the estimate:*

(1) *for any $k \geq 0$ and $1 < q < \infty$, there exists a C such that*

$$\|\tilde{u}\|_{W_k^q(B_1(0))} \leq C \|u\|_{p; B_1^+(0) \setminus B_{\frac{1}{2}}(0)} \quad (37)$$

(2) *if $c \geq 0$, then for $1 < q < \infty$, there exists a C such that*

$$\|u\|_{1,q; B_1(0)} \leq C \|u\|_{p; B_1^+(0) \setminus B_{\frac{1}{2}}(0)} \quad (38)$$

(3) *if $c < 0$, then for $1 < q < \frac{2}{q_j}$, there exists a constant C such that the above inequality in (2) holds.*

Proof. Because the original proof of Lemma 4.4 in [10] viewed the constant c as a unity again, so it is obvious. \square

Combining Lemma 3.2 and Lemma 3.3, we have theorems in case of the cylindrical metric corresponding to Corollary 4.5 and Lemma 4.6 in [10]:

Lemma 3.4. *If $c > 0$ at $z_l = 0$, then for $1 < p < 2/(1 - \bar{\delta}_j)$, where $\bar{\delta}_j = \min_{l: c_{jl} > 0}(c_{jl})$, there is*

$$\|u\|_{1,p; B_1(0)} + \left\| \frac{u}{z} \right\|_{p; B_1(0)} \leq C \|u\|_{1,p; B_1(0)}$$

Proof. Use the same argument in the proof above again. \square

For $c = a_j(h_l) = 0$, we have similar estimates (Note that $c = a_j(h_l) \leq 0$ in Lemma 4.6 of [10])

Lemma 3.5. *If $c = a_j(z_l) = 0$, then for $1 < p < \infty$ and any $u = \tilde{u}e_j \in L_1^p(B_1(0), |L_j|)$ satisfying $u(0) = 0$, there is*

$$\|u\|_{1,p; B_1(0)} + \left\| \frac{u}{z} \right\|_{p; B_1(0)} \leq C \|u\|_{1,p; B_1(0)} \quad (39)$$

Proof. It is the same as the proof of Lemma 4.6 in [10]. \square

Next we have regularity of local solution.

Lemma 3.6. *Let $\bar{\partial}u = f$ in $B_1^+(0)$, where $u \in L_1^p(B_1^+(0), |L_j|)$ and $f \in L_1^p(B_1^+(0), |L_j| \otimes \wedge^{0,1})$, then $u \in L_1^p(B_1^+(0), |L_j|)$ and the inequality*

$$\|u\|_{1,p;B_1(0)} \leq C(\|u\|_{p;B_1^+(0) \setminus B_{\frac{1}{2}}(0)} + \|f\|_{p;B_1^+(0)}) \quad (40)$$

holds if the following two conditions are satisfied:

- $a_{j,l} = a_j(h_l) - q_j + 2/p \in \mathbb{R} \setminus \mathbb{Z}$
- $c \geq 0, 1 < p < \infty$

Proof. The proof is similar to the proof of Lemma 4.7 in [10]. Under the assumptions on parameters c and p , one has

$$\begin{aligned} \|u\|_{1,p;B_1(0)} &\leq \|u - u_s\|_{1,p;B_1(0)} + \|u_s\|_{1,p} \\ &\leq C(\|u - u_s\|_{p;B_1^+(0) \setminus B_{\frac{1}{2}}(0)} + \|f\|_p) \\ &\leq C(\|u\|_{p;B_1^+(0) \setminus B_{\frac{1}{2}}(0)} + \|u_s\|_p + \|f\|_p) \\ &\leq C(\|u\|_{p;B_1^+(0) \setminus B_{\frac{1}{2}}(0)} + \|f\|_{p;B_1^+(0)}) \end{aligned}$$

where the second inequality comes from (2) of Lemma 3.2 and (1) of Lemma 3.3, and the fourth inequality comes from Lemma 3.2. \square

Now by the above lemma, we can obtain the following global estimate.

Lemma 3.7. *Let $\bar{\partial}u = f$ on Σ , where $u \in L_1^p(\Sigma, |L_j|)$ and $f \in L_1^p(\Sigma, |L_j| \otimes \wedge^{0,1})$. Then $u \in L_1^p(\Sigma, |L_j|)$ and the inequality*

$$\|u\|_{1,p} \leq C(\|u\|_{L^p(\Sigma \setminus \cup_{l=1}^k B_{\frac{1}{2}}(z_l))} + \|\bar{\partial}u\|_p) \quad (41)$$

holds if the following two conditions are satisfied:

- $a_{j,l} = a_j(h_l) + 2/p \in \mathbb{R} \setminus \mathbb{Z}$ for any $l = 1, \dots, k$
- $c \geq 0, 1 < p < \infty$

Proof. It is totally similar to Lemma 4.8 in [10]. \square

Next we can similarly generalize the index decomposition theorem (Theorem 2.1) to be in case of the cylindrical metric.

Theorem 3.8. *For the cylindrical metric, we have*

$$\begin{aligned} (1) & \text{ind}(\bar{\partial}^{t,\theta} : \hat{W}_{1,1+\kappa}^p \rightarrow W_{0,1+\kappa}^p) \\ &= \text{ind}(\bar{\partial}^{t,\theta} : W_1^{p,B}(\text{inn}) \rightarrow W_0^p(\text{inn})) + \sum_{l=1}^k \text{ind}(\bar{\partial}^{t,\theta} : \hat{W}_{1,1+\kappa_{j,l}}^{p,B} \rightarrow W_{0,1+\kappa_{j,l}}^p) \\ (2) & \text{ind}(\bar{\partial}^{t,\theta} : W_{1,1+\kappa}^p \rightarrow W_{0,1+\kappa}^p) \\ &= \text{ind}(\bar{\partial}^{t,\theta} : W_1^{p,B}(\text{inn}) \rightarrow W_0^p(\text{inn})) + \sum_{l=1}^k \text{ind}(\bar{\partial}^{t,\theta} : W_{1,1+\kappa_{j,l}}^{p,B} \rightarrow W_{0,1+\kappa_{j,l}}^p) \end{aligned}$$

Now we can prove Theorem 3.1.

Proof of Theorem 3.1. We will not repeat the same parts as Theorem 4.10 in [10], and only give the different parts. Because $c = c_{jl} = a_j(z_l) \geq 0$, we only need make a change related to $c = c_{jl} < 0$. The most important step is as follows:

In case of the smooth metric, when $c = c_{j,l} < 0$, we have $0 < 1 + k_{j,l} < 1$ if $2 < p < 2/q_j$, where $k_{j,l} = -c_{j,l} - 2/p$, then we deduce $W_{0,1+k}^p \subset W_0^2$.

In case of the cylindrical metric, only $c = c_{jl} = 0$ can happen, so we have $0 < 1 + k_{j,l} < 1$ only simplifying the condition as $p > 2$.

Later process is totally the same as the proof of Theorem 4.10 in [10]. \square

Further, we can similarly get the index transformation theorem with weights and we omit the proof.

Theorem 3.9. (*Index transformation theorem with weights*) Under the assumption of Theorem 3.1, if $p > 2$, we have the relation

$$\begin{aligned} & \text{ind}(\bar{\partial}_j^\delta : L_1^{p,\delta}(\Sigma, |L_j|) \rightarrow L^{p,\delta}(\Sigma, |L_j| \otimes \wedge^{0,1})) \\ &= \text{ind}(\bar{\partial}_j^{t,\theta,\delta} : W_{1,1+\kappa+\delta}^p \rightarrow W_{0,1+\kappa+\delta}^p) + \#\{z_l : c_{jl} = 0\}. \end{aligned}$$

In the following computation in case of the cylindrical metric, we can directly apply Theorem 3.9 to prove Theorem 1.4.

3.2 Index jumping formula

First by the index transformation theorem with weights (Theorem 3.9) above, we get :

$$\begin{aligned} & \text{ind}(\bar{\partial}_j^\delta : L_1^{p,\delta}(\Sigma, |L_j|) \rightarrow L^{p,\delta}(\Sigma, |L_j| \otimes \wedge^{0,1})) \\ &= \text{ind}(\bar{\partial}_j^{t,\theta,\delta} : W_{1,1+\kappa+\delta}^p \rightarrow W_{0,1+\kappa+\delta}^p) + \#\{z_l : c_{jl} = 0\} \end{aligned}$$

We can focus on the $\text{ind}(\bar{\partial}_j^{t,\theta,\delta} : W_{1,1+\kappa+\delta}^p \rightarrow W_{0,1+\kappa+\delta}^p)$. Notice that here the Sobolev spaces $W_{1,1+\kappa+\delta}^p, W_{0,1+\kappa+\delta}^p$ are normal Sobolev spaces, so now we can apply Lockhart-McOwen theory and immediately get

Theorem 3.10. Let $\bar{\partial}_j^{t,\theta,\delta} : W_{1,1+k+\delta}^p \rightarrow W_{0,1+k+\delta}^p$ be as above, $\forall \delta_1 < \delta_2 \in \mathbb{R} \setminus \mathbb{Z}$, then we have following index jumping formula:

$$\text{ind}(\bar{\partial}_j^{t,\theta,\delta_1}) - \text{ind}(\bar{\partial}_j^{t,\theta,\delta_2}) = N(\delta_1, \delta_2) = [\delta_2] - [\delta_1]$$

Proof. Directly apply Theorem 2.8 to the operators $\bar{\partial}^{t,\theta,\delta_1}, \bar{\partial}^{t,\theta,\delta_2}$. \square

Let $k = 1$, which means that we have only one marked point, by Theorem 3.9, Theorem 3.10, we have

Theorem 3.11.

$$\text{ind}(\bar{\partial}_j^{\delta_1}) - \text{ind}(\bar{\partial}_j^{\delta_2}) = [\delta_2] - [\delta_1] \quad (42)$$

Assume the action of orbifold line bundle L_j at marked point z_l is $a_j(h_l)(l = 1, \dots, k)$, let $k_{j,l} = -a_j(h_l) - 2/p$, $k = (k_{j,1}, \dots, k_{j,k})$ as [10]. Consider corresponding weight vector $\delta = (\delta_{j,1}, \dots, \delta_{j,k})$, $\delta' = (\delta'_{j,1}, \dots, \delta'_{j,k})$ and weighted Sobolev space $W_{s,1+k+\delta}^p, \hat{W}_{s,1+k+\delta}^p$

By Theorem 3.8, which can transform the total index jumping into the sum of computation of each end, Theorem 3.9 and Theorem 3.11, we have

Theorem 3.12. *In case of the cylindrical metric, we have an index jumping formula:*

$$\text{ind}(\bar{\partial}_j^\delta) - \text{ind}(\bar{\partial}_j^{\delta'}) = \sum_{l=1}^k ([\delta'_{j,l}] - [\delta_{j,l}]) \quad (43)$$

Now we can apply results above to prove Theorem 1.4.

Proof of Theorem 1.4. Consider the linearized operator of the Witten map

$$D^\delta = D_{\varphi,\mu} WI : L_1^{p,\delta}(\Sigma, L_1 \times L_2 \times \dots \times L_N) \rightarrow L^{p,\delta}(\Sigma, L_1 \otimes \wedge^{0,1}) \times \dots \times L^{p,\delta}(\Sigma, L_N \otimes \wedge^{0,1})$$

Note that D^δ is an operator over orbicurve, [12] has given the index relation between D^δ and $\bar{\partial}_j^{orb,\delta} : L_1^{p,\delta}(\Sigma, L_j) \rightarrow L^{p,\delta}(\Sigma, L_j \otimes \wedge^{0,1})$. In fact they got :

$$\text{ind}(D^\delta) = \sum_{j=1}^N \text{ind}(\bar{\partial}_j^{orb,\delta}) - \sum_{l=1}^k \sum_{j:a_j(h_l)=0} 1, \quad (44)$$

where $\bar{\partial}_j^{orb,\delta} : L_1^{p,\delta}(\Sigma, L_j) \rightarrow L^{p,\delta}(\Sigma, L_j \otimes \wedge^{0,1})$ is an operator on orbifold line bundle L_j , it is different from the operator $\bar{\partial}_j^\delta$ on desingularization $|L_j|$ of L_j . However, we have

Theorem 3.13.

$$\text{ind}(\bar{\partial}_j^{orb,\delta}) = \text{ind}(\bar{\partial}_j^\delta) \quad (45)$$

Proof. Directly apply Proposition 4.2.2 of [8] to our case. \square

Let $E = L_j(j = 1, \dots, N)$, where N stands for the number of variable of quasi-homogeneous polynomial W in Spin equations, then by (45) we have

$$\text{ind}(\bar{\partial}_j^{orb,\delta}) = \text{ind}(\bar{\partial}_j^\delta)$$

Let $\delta = (\delta_{jl}), \delta' = (\delta'_{jl})$ be weight matrix($j = 1, \dots, N; l = 1, \dots, k$), and assume all weight component $\delta_{jl} \in \mathbb{R} \setminus \mathbb{Z}, \delta'_{jl} \in \mathbb{R} \setminus \mathbb{Z}$.

Combining (43), (44) and (45), we complete the proof of Theorem 1.4. \square

4 Appendix

4.1 Boundary Maslov index

In this part, we mainly recall the definition and some properties of boundary Maslov index (see [14] for more details).

According to [14], first we need a special decomposition of base Riemannian surface. Let's recall some definitions.

Definition 4.1. A *decomposition* of a 2-manifold Σ_{02} is a pair of submanifolds Σ_{01}, Σ_{12} of Σ_{02} such that $\Sigma_{02} = \Sigma_{01} \cup \Sigma_{12}$, $\Sigma_{01} \cap \Sigma_{12} = \partial\Sigma_{01} \cap \partial\Sigma_{12}$. It follows that $\partial\Sigma_{ij} = \Gamma_i \cup \Gamma_j$, $\Gamma_i \cap \Gamma_j = \emptyset$, where Γ_i is a disjoint union of circles in Σ_{02} and $\Gamma_1 = \Sigma_{01} \cap \Sigma_{12}$.

Definition 4.2. A bundle pair (E, F) over Σ consists of a complex vector bundle $E \rightarrow \Sigma$ and a total real subbundle $F \subset E|_{\partial\Sigma}$ over the boundary. A *decomposition* of a bundle pair (E_{02}, F_{02}) over Σ_{02} consists of two bundle pairs, $(E_{01}, F_0 \cup F_1)$ over Σ_{01} and $(E_{12}, F_1 \cup F_2)$ over Σ_{12} , such that Σ_{01}, Σ_{12} is a decomposition of Σ_{02} as in definition 4.1 and $F_i \subset E_{02}|_{\Gamma_i}$. Next we list the axiomatic definitions of boundary Maslov index:

Theorem 4.1. *There is a unique operation, called boundary Maslov index, that assigns an integer $\mu(E, F)$ to each bundle pair (E, F) and satisfies the following axioms:*

(Isomorphism): If $\Phi : E_1 \rightarrow E_2$ is a vector bundle isomorphism covering a diffeomorphism $\phi : \Sigma_1 \rightarrow \Sigma_2$, then

$$\mu(E_1, F_1) = \mu(E_2, \Phi(F_1))$$

(Direct sum): $\mu(E_1 \oplus E_2, F_1 \oplus F_2) = \mu(E_1, F_1) + \mu(E_2, F_2)$

(Composition): For a composition of a bundle pair decomposition as in definition 4.2, we have

$$\mu(E_{02}, F_{02}) = \mu(E_{01}, F_{01}) + \mu(E_{12}, F_{12})$$

(Normalization): For $\Sigma = D$ the unit disc, $E = D \times \mathbb{C}$ the trivial bundle and $F_z = e^{ik\theta/2}\mathbb{R}$ for $z = e^{i\theta} \in \partial D = S^1$ we have

$$\mu(D \times \mathbb{C}, F) = k$$

In addition, we have another important property which relates the boundary Maslov index to the first Chern class.

Theorem 4.2. *If $\partial\Sigma = \emptyset$, we have*

$$\mu(E, \emptyset) = 2 < c_1(E), [\Sigma] >$$

where $c_1(E)$ is the first Chern class of E and $[\Sigma]$ is the fundamental class.

4.2 Donaldson index theory

We adopt notations of [9] in the following discussion. Consider a non-compact Riemann manifold $X = Y \times \mathbb{R}$ called a tubular end, where Y is a compact 3-dimensional manifold. Let $\pi : Y \times \mathbb{R} \rightarrow Y$ be the projection, $P \rightarrow Y$ a G -principal bundle, where G is a compact Lie group, π^*P pulled bundle on X , A is a connection on P , we write A as pulled connection on π^*P if no confusion. Assume d_A is the corresponding covariant derivative of A , d_A^* is its adjoint operator.

Donaldson defined an operator which is very important for his theory (see [9]) as follows:

$$D_A = -d_A^* \bigoplus d_A^+ : \Omega_X^1(\mathfrak{g}_P) \rightarrow \Omega_X^0(\mathfrak{g}_P) \bigoplus \Omega_X^1(\mathfrak{g}_P) \quad (46)$$

where \mathfrak{g}_P stands for adjoint bundle of principal bundle π^*P associated to the adjoint representation of G , $\Omega_X^k(\mathfrak{g}_P)$ is smooth k -forms taking values in \mathfrak{g}_P . What's more, Donaldson transformed D_A to be a simple form as follows:

$$D_A = \frac{d}{dt} + L : \Omega_X^0(\mathfrak{g}_P) \bigoplus \Omega_X^1(\mathfrak{g}_P) \rightarrow \Omega_X^0(\mathfrak{g}_P) \bigoplus \Omega_X^1(\mathfrak{g}_P) \quad (47)$$

where L is a self dual elliptic operator. This expression is very important to his theory, that is to say, when L is invertible, by separation of variables he proved the following theorem:

Theorem 4.3. $D_A = \frac{d}{dt} + L : L_1^2 \rightarrow L^2$ is a Fredholm operator.

Then Donaldson studied the following index problem. Assume that Riemannian manifold X is a disjoint union of two disconnected components $X = X_1 \cup X_2$, and two ends $Y \times (0, \infty), \bar{Y} \times (0, \infty)$ which are identified are contained in different components.

Now we consider a family of Riemannian manifolds $X^{\sharp(T)}$, depending on a real parameter $T > 0$, obtained by identifying the two ends of X . For fixed T we first delete the infinite portions $Y \times [2T, \infty), \bar{Y} \times [2T, \infty)$ from the two ends, and then identify $(y, t) \in Y \times (0, T) \subset X_1$ with $(y, 2T - t) \in Y \times (T, 2T) \subset X_2$.

This gives a connected compact Riemann manifold $X^{\sharp(T)}$. Clearly these are all diffeomorphic for different values of T . We will denote the manifold by X^{\sharp} when the T dependence is not important. The procedure is a generalization of the connected sum operation on manifolds.

Fix an isometry between Y and \bar{Y} . Suppose there are vector bundles E_1, E_2 on X_1, X_2 respectively. Consider smooth, compactly supported section spaces $\Gamma_c(E_1), \Gamma_c(E_2)$ of E_1, E_2 , we define Sobolev norm as follows:

$$\|f_i\|_{L^p} : = \left(\int_{X_i} |f_i|^p d\text{vol} \right)^{1/p}, \forall f_i \in \Gamma_c(E_i) \quad (48)$$

$$\|\rho_i\|_{L_1^p} : = \left(\int_{X_i} (|\rho_i|^p + |D_A \rho_i|^p) d\text{vol} \right)^{1/p}, \forall \rho_i \in \Gamma_c(E_i) \quad (49)$$

where $|\cdot|$ denote norm induced by a Riemann metric g .

Let $L^p(E_i), L_1^p(E_i)$ be Sobolev spaces by completing spaces $\Gamma_c(E_1), \Gamma_c(E_2)$ respectively under the form of $\|\cdot\|_{L^p}, \|\cdot\|_{L_1^p}$

Consider differential operators D_i acting on Sobolev spaces $L_1^p(E_i), i = 1, 2$. Donaldson proved these operators were Fredholm operators in [9], so $\text{ind } D_i$ are well defined.

There is an obvious way of constructing an bundle $E^{\sharp(T)}$ over $X^{\sharp(T)}$, identifying the bundles over the ends, and the operator $D^{\sharp(T)}$ over $X^{\sharp(T)}$. We write $D^{\sharp(T)}$ as D^{\sharp} when the T dependence is not important.

Theorem 4.4 (L^p -Index gluing theorem).

$$\text{ind}(D^{\sharp}) = \text{ind}(D_1) + \text{ind}(D_2) \quad (50)$$

Two theorems above are proved under norms L^2 and L_1^2 , but Donaldson also pointed out these two theorems also can be extended under norms L^p and L_1^p . Theorem 4.4 is almost the same as Theorem 2.5, so next we only give the proof of Theorem 2.5.

Proof of Theorem 2.5. We will imitate Donaldson's proof of L^2 -edition step by step. The proof involves four steps, for the first three steps we suppose that the operators D^i over X_i have zero cokernel, then admit bounded right inverses

$$Q_i : L^p = L^p(E_i) \rightarrow L_1^p = L_1^p(E_i)$$

by a following lemma, with $\|Q_i(\rho)\|_p = \|Q_i(\rho)\|_{L^p} \leq C_p \|\rho\|_p$ and $D_i Q_i = 1$.

Lemma 4.5. *Assume L is invertible. Consider $D = \frac{d}{dt} + L : L_1^p \rightarrow L^p$ has zero cokernel, then there exists a bounded right inverse operator Q and a constant C_p such that*

$$\|Q\rho\|_{L^p} \leq C_p \|\rho\|_{L^p}, DQ = 1$$

where ρ is any smooth compactly supported sections of vector bundle E_i .

In L^2 case, we use separation of variables because function space is Hilbert space. When facing L^p case, we can not use separation of variables again. Fortunately we can use the expression of inversion operator Q in L^2 case to construct the inversion operator of L^p edition as follows:

$$Q(\rho) = \int_{-\infty}^{\infty} K(s-t)\rho(t)dt$$

where the kernel K is an operator-valued function.

Step 1 We construct, for large T , an injection

$$\alpha : \ker D^{\sharp} \rightarrow \ker D_1 \oplus \ker D_2$$

In fact we construct a map which is close to being an isometric embedding, with respect to the metrics on the kernels induced by the L^2 norms. To do this we fix functions ϕ_1, ϕ_2 on $\Sigma^{\sharp(T)}$ such that $\phi_1^p + \phi_2^p = 1$, with ϕ_i supported in $X_i(3T/2)$ and such that $\|\nabla \phi_i\|_{L^\infty} = \epsilon(T)$, where $\epsilon(T) \rightarrow 0$ as $T \rightarrow \infty$. It is easy to write down such functions, indeed we can obviously take $\epsilon(T) = \text{const.} T^{-1}$. We then put, for $f \in \ker D^\sharp$, $\alpha(f) = (f_1, f_2)$ where

$$f_i = \phi_i f - Q_i D_i(\phi_i f).$$

Here we are regarding $\phi_i f$ as being defined over Σ_i in the obvious way, using the fact that ϕ_i is supported in $\Sigma_i(2T)$. The section f_i lies in the kernel of D_i , since Q_i is a right inverse. It is also a small perturbation of $\phi_i f$ in that we have

$$\begin{aligned} \|f_i - \phi_i f\|_p &= \|Q_i D_i(\phi_i f)\|_p \\ &\leq C_p \|D_i(\phi_i f)\|_p \\ &= C_p \|\nabla \phi_i f\|_p \\ &\leq C_p \|\nabla \phi_i\|_p \|f\|_p \\ &\leq C_p \epsilon(T) \|f\|_p. \end{aligned}$$

Here we have used the fact that $D^\sharp f = 0$, and that D_i can be identified with D^\sharp over the support of ϕ_i .

To complete the first step we now observe that, for any f on $\Sigma^{\sharp(T)}$,

$$\|\phi_1 f\|_p^p + \|\phi_2 f\|_p^p = \|f\|_p^p,$$

since $\phi_1^p + \phi_2^p = 1$. Otherwise said, the map $f \mapsto (\phi_1 f, \phi_2 f)$ defines an isometric embedding of $L_{\Sigma^{\sharp(T)}}^p$ in $L_{\Sigma_1}^p \oplus L_{\Sigma_2}^p$. This means that α is approximately an isometry for large T , precisely,

$$\begin{aligned} \|\alpha(f)\|_p - \|f\|_p &= \|\alpha(f)\|_p - \|(\phi_1 f, \phi_2 f)\|_p \\ &\leq \|(f_1 - \phi_1 f, f_2 - \phi_2 f)\|_p \\ &\leq \sqrt[p]{2} C_p \epsilon(T) \|f\|_p \end{aligned}$$

where we used estimate above and the definition of norm that $\|(f_1, f_2)\|_p^p := \|f_1\|_p^p + \|f_2\|_p^p$, so α is injective once T is sufficient large such that

$$\epsilon(T) < \frac{1}{\sqrt[p]{2} C_p}.$$

Step 2 We show that, under the same assumption of the existence of the right inverses Q_i , the operator D^\sharp is also surjective for large T . To do this it suffices to construct a map $P : L^p \rightarrow L_1^p$ over $\Sigma^{\sharp(T)}$ such that

$$\|D^\sharp P(\rho) - \rho\|_p \leq k \|\rho\|_p$$

where $k < 1$. For then the operator $PD^\sharp - 1$ is invertible and $Q = P(PD^\sharp - 1)^{-1}$ is a right inverse for D^\sharp . We construct P by splicing together the operators Q_i over the individual manifolds. Write $\beta_i = \phi_i^p$, where ϕ_i are the cut-off functions above. Thus $\beta_1 + \beta_2 = 1$. And since $0 \leq \phi_i \leq 1$, the gradient of $\beta_i \leq 2\epsilon(T)$. We define

$$P(\rho) = \beta_1 Q_1(\rho_1) + \beta_2 Q_2(\rho_2)$$

over $\Sigma^{\sharp(T)}$, where ρ_i is the restriction of ρ to $\Sigma_i(2T) \subset \Sigma^{\sharp(T)}$, extended by zero over the remainder of X_i . Similarly, $\beta_i Q_i(\rho_i)$ is regarded as a section over $\Sigma^{\sharp(T)}$, extending by zero outside the support of β_i . Then

$$D^\sharp P(\rho) = (\beta_1 D^\sharp Q_1(\rho_1) + \beta_2 D^\sharp Q_2(\rho_2)) + ((\nabla \beta_1) * Q_1(\rho_1) + (\nabla \beta_2) * Q_2(\rho_2))$$

Here, as before, we use $*$ to denote a certain algebraic operation. Now $\beta_i D^\sharp Q_i(\rho_i) = \beta_i \rho_i = \beta_i \rho$, since we can identify D^\sharp with D_i and ρ_i with ρ over the support of β_i . So the first two terms in the expression above yield ρ and the remainder has norm bounded by

$$\sum ||(\nabla \beta_i) * Q_i(\rho_i)||_p \leq 4C_p \epsilon(T) ||\rho||_p$$

So

$$||D^\sharp P(\rho) - \rho||_p \leq 4C_p \epsilon(T) ||\rho||_p$$

and we achieve the desired "approximate inverse" by taking T so large that $\epsilon(T) \leq 1/(4C_p)$. This completes the second step in the proof.

Step 3 In the third step we construct, under the same assumption of the surjectivity of D_i , a linear injection $\alpha' : \ker D_1 \oplus \ker D_2 \rightarrow \ker D^\sharp$ for large enough T . For this we first return to the construction of the operator Q above, and note that it admit an L^2 bound:

$$||Q(\rho)||_p \leq C_p ||\rho||_p$$

say, for all large enough values of T . With this observation the map α' can be constructed in a similar fashion to the map α in the first step.

For elements f_i of the kernel of the D_i over Σ_i we set $\alpha'(f_1, f_2) = g - QD^\sharp g$, where $g = \beta_1 f_1 + \beta_2 f_2$

Here we have identified appropriate sections over X_i and X , in the way which will now be familiar to the reader. Just as in the first step, we see that the L^2 norm of the 'correction term' $QD^\sharp g$ is bounded by an arbitrarily

small multiple of $\|(f_1, f_2)\|_p$. In fact,

$$\begin{aligned}
\|QD^\sharp g\|_p^p &= \|QD^\sharp(\beta_1 f_1 + \beta_2 f_2)\|_p^p \\
&\leq C_p^p \|D^\sharp(\beta_1 f_1 + \beta_2 f_2)\|_p^p \\
&\leq C_p^p (\|\nabla \beta_1\|_p^p + \|\nabla \beta_2\|_p^p) \\
&\leq (pC_p \epsilon(T))^p (\|f_1\|_p^p + \|f_2\|_p^p) \\
&= (pC_p \epsilon(T))^p \|(f_1, f_2)\|_p^p
\end{aligned}$$

Here we use again two facts, one is that $D_i f_i = 0$ and that D^\sharp can be identified with D_i over the support of β_i , another is that $\|(f_1, f_2)\|_p^p := \|f_1\|_p^p + \|f_2\|_p^p$. It remains only to show that the L^p norm of g is close to that of (f_1, f_2) for large T .

By integrability of f_i , for any $\eta = \eta(T) > 0$, we can choose sufficient large T_0 such that for $T \geq T_0$ then,

$$\int_{(T/2, \infty)} |f_i|^p \leq \eta \|f_i\|_p^p, \forall f_i \in \ker D_i.$$

Since $\beta_i = 1$ on the segment $Y \times (0, T/2)$ of the tube in Σ_i , we clearly have

$$\begin{aligned}
\|g\|_p^p &= \|\beta_1 f_1 + \beta_2 f_2\|_p^p \geq \int_{Y \times (0, T/2)} |f_1|^p + \int_{\bar{Y} \times (0, T/2)} |f_2|^p \\
&= \|f_1\|_p^p + \|f_2\|_p^p - \int_{Y \times (T/2, \infty)} |f_1|^p + \int_{\bar{Y} \times (T/2, \infty)} |f_2|^p \\
&\geq (1 - \eta)(\|f_1\|_p^p + \|f_2\|_p^p) \\
&= (1 - \eta)\|(f_1, f_2)\|_p^p.
\end{aligned}$$

And obviously $\|g\|_p^p = \|\beta_1 f_1 + \beta_2 f_2\|_p^p \leq \|(f_1, f_2)\|_p^p$, so we have

$$\begin{aligned}
| \|(f_1, f_2)\|_p^p - \|\alpha'(f_1, f_2)\|_p^p | &= | \|(f_1, f_2)\|_p^p - \|g - QD^\sharp g\|_p^p | \\
&\leq | \|(f_1, f_2)\|_p^p - (\|g\|_p^p - \|QD^\sharp g\|_p^p) | \\
&\leq (\eta(T) + (pC_p \epsilon(T))^p) \|(f_1, f_2)\|_p^p
\end{aligned}$$

Therefore, when $\alpha'(f_1, f_2) = 0$, we can choose sufficient large T such that $(\eta(T) + (pC_p \epsilon(T))^p) < 1$, then by inequality above, we get $\|(f_1, f_2)\|_p^p = 0 \Rightarrow (f_1, f_2) = (0, 0)$, so we prove that α' is an injection.

These first three steps complete the proof of the "gluing formula" in the case when the D_i are surjective. For in this case we have, by step 2, $\text{ind}(D_i) = \dim \ker D_i$, $\text{ind}(D^\sharp) = \dim \ker D^\sharp$ for large T . By step 1, $\dim \ker D^\sharp \leq \dim \ker D_1 + \dim \ker D_2$, and step 3 gives the reverse inequality, so $\dim \ker D^\sharp = \dim \ker D_1 + \dim \ker D_2$ as required. Hence

$$\text{ind}(D^\sharp) = \text{ind}(D_1) + \text{ind}(D_2)$$

Step 4 This step is totally the same as Donaldson's proof of L^2 -edition. First we remove the assumption that the operators D_i are surjective. We do this by modifying the operators. Assume that $n_i = \dim \text{coker } D_i$, this is well defined because D_i are Fredholm operators. We can choose injective maps $U_i : R^{n_i} \rightarrow \Gamma(E_i)$ with images supported in the interior of the Σ_i , and such that

$$\tilde{D}_i \equiv D_i \oplus U_i : \Gamma(E_i) \oplus R^{n_i} \rightarrow \Gamma(E_i)$$

is surjective. We have

$$\text{ind} \tilde{D}_i = \text{ind } D_i + \dim \text{coker } D_i = \text{ind } D_i + n_i$$

We can form an obvious operator $\tilde{D}^\# = D^\# \oplus U_1 \oplus U_2$ over $\Sigma^{\#(T)}$, and the proof above goes without any change to show that $\text{ind} \tilde{D}^\# = \text{ind} \tilde{D}_1 + \text{ind} \tilde{D}_2$, so

$$\begin{aligned} \text{ind} D^\# &= \text{ind} \tilde{D}^\# - (n_1 + n_2) = \text{ind} \tilde{D}_1 + \text{ind} \tilde{D}_2 - (n_1 + n_2) \\ &= (\text{ind} \tilde{D}_1 - n_1) + (\text{ind} \tilde{D}_2 - n_2) = \text{ind} D_1 + \text{ind} D_2 \end{aligned}$$

We ultimately complete the proof of Theorem 2.5. \square

When L is not invertible, Donaldson considered weight Sobolev space $L^{2,\alpha}$ and $L_1^{2,\alpha}$, which are defined as follows:

$$\|f\|_{L_1^{2,\alpha}} = \|e^{\alpha t} f\|_{L_1^2}, \|f\|_{L^{2,\alpha}} = \|e^{\alpha t} f\|_{L^2} \quad (51)$$

where $\alpha \in \mathbb{R}$. Then consider $D_A = \frac{d}{dt} + L : L_1^{2,\alpha} \rightarrow L^{2,\alpha}$, because multiple operation of $e^{\alpha t}$ is an isometry from $L_1^{2,\alpha}$ and $L^{2,\alpha}$ to L_1^2 and L^2 separately. Therefore, $D_A = \frac{d}{dt} + L : L_1^{2,\alpha} \rightarrow L^{2,\alpha}$ is equivalent to:

$$e^{\alpha t} D_A e^{-\alpha t} = \frac{d}{dt} + (L - \alpha) : L_1^2 \rightarrow L^2 \quad (52)$$

We can see the introduction of weighted Sobolev space is equivalent to replace the operator L of $(L - \alpha)$, then theorems without weights above can be easily to generalized in weighted Sobolev space.

Now let's consider the gluing problem with weights. Suppose we are in the situation above, we consider $D_A = D + \nu : \frac{d}{dt} + L : L_1^{p,\alpha} \rightarrow L^{p,\alpha}$ where ν is an algebraic operator, represented as multiplication by $\alpha_i, i = 1, 2$ in our description over the i th end. The operator D is represented as $\frac{d}{dt_1} + L_Y$ over the first end and as $\frac{d}{dt_2} + L_{\bar{Y}}$ over the second. When we identify the ends to form $\Sigma^\#$ we reverse the time co-ordinates, so $\frac{d}{dt_1}$ corresponds to $-\frac{d}{dt_2}$,

and this marries up with the natural identification $L_Y = -L_{\bar{Y}}$. Thus, in our gluing operation, the operator

$$\frac{d}{dt_1} + L_Y + \alpha_1$$

over the first end can naturally be identified with

$$\frac{d}{dt_2} + L_{\bar{Y}} + \alpha_2$$

over the second if $\alpha_2 = -\alpha_1$, now we still write D^\sharp as the operator after gluing.

In this case the arguments we used before go through without any change to show that

Theorem 4.6. *Assuming $\alpha \in \mathbb{R}$ such that $L - \alpha$ is invertible, we have*

$$\text{ind}(D^\sharp) = \text{ind}(D^{\alpha_1}) + \text{ind}(D^{-\alpha_1})$$

We can further consider weight vector case which corresponds to more ends. Choose a weights α_i for each end $Y_i \times (0, \infty)$ of Σ . Fix a positive function W on Σ which is equal to $e^{\alpha_i t}$ on the i th end and define norms:

$$\|f\|_{L^{p,\vec{\alpha}}} = \|Wf\|_{L^p}, \|f\|_{L_1^{p,\vec{\alpha}}} = \|Wf\|_{L_1^p}$$

with completions $L^{p,\vec{\alpha}}, L_1^{p,\vec{\alpha}}$. Different choices of W , with the same weight vector $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_N)$, give equivalent norms.

Similarly, we can easily obtain index gluing formula in weight vector case.

Theorem 4.7. *Assuming $\vec{\alpha} \in \mathbb{R}^N$ such that $L - \vec{\alpha}$ is invertible, we have*

$$\text{ind}(D^\sharp; (\alpha_2, \dots, \alpha_N)) = \text{ind}(D^{(\alpha_1, \alpha_2, \dots, \alpha_N)}) + \text{ind}(D^{(-\alpha_1, \alpha_2, \dots, \alpha_N)})$$

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